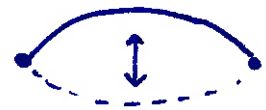
Wave Equation (d'Alembert solution)

we have been looking at standing waves 

but that can also be viewed as a combination of two traveling waves moving in opposite directions

we can see this in the Fourier series solution:

Problem A :
$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

we know $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

rewrite the solution

$$y(x,t) = \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x+at)\right)}_{\text{LEFT moving wave, magnitude } \frac{1}{2}A_n} + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L}(x-at)\right)}_{\text{same but moving RIGHT}}$$

d'Alembert solution was from 1747 (60 years before Fourier)

$y_{tt} = a^2 y_{xx}$ $\qquad \underbrace{-\infty < x < \infty}_{\text{no BCs}} \qquad t > 0$

ICs: $y(x,0) = f(x)$ initial displacement

$y_t(x,0) = g(x)$ initial velocity

d'Alembert: observer moving w/ a wave would see a wave that never changes

→ use that as the coordinate system

follow wave moving right: $\dfrac{dx}{dt} = a \rightarrow x - at = \text{constant}$

left: $\dfrac{dx}{dt} = -a \rightarrow x + at = \text{constant}$

let $\left.\begin{aligned} \xi &= x + at \\ \eta &= x - at \end{aligned}\right\}$ rewrite $y_{tt} = a^2 y_{xx}$

$$y_x = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial x} \rightarrow y_x = y_\xi + y_\eta$$

$$y_t = a(y_\xi - y_\eta)$$

$$y_{xx} = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta}$$

$$y_{tt} = a^2(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta})$$

Sub into $y_{tt} = a^2 y_{xx}$

we get $\boxed{y_{\xi\eta} = 0}$

integrate with respect to ξ : $y_\eta(\xi,\eta) = \phi(\eta)$

again : $y(\xi,\eta) = \phi(\eta) + \psi(\xi)$

back to x and t

$$\boxed{y(x,t) = \phi(x-at) + \psi(x+at)}$$

ICs: $y(x,0) = f(x)$ displacement $\qquad y(x,t) = \phi(x-at) + \psi(x+at)$

$y_t(x,0) = g(x)$ velocity

$$\boxed{f(x) = \phi(x) + \psi(x)}$$

$$y_t(x,t) = \frac{\partial \phi}{\partial(x-at)} \frac{\partial(x-at)}{\partial t} + \frac{\partial \psi}{\partial(x+at)} \frac{\partial(x+at)}{\partial t}$$

at $t=0$

$$g(x) = \phi'(x) \cdot -a + \psi'(x) \cdot a$$

or $\boxed{g(x) = -a\phi'(x) + a\psi'(x)}$

integrate from x_0 to x

$$\boxed{\int_{x_0}^{x} g(s)\,ds = -a\phi(x) + a\psi(x)}$$

Solve 1st and 3rd eqs simultaneously

⋮

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2a} \int_{x_0}^{x} g(s)\,ds$$

$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2a} \int_{x_0}^{x} g(s)\,ds$$

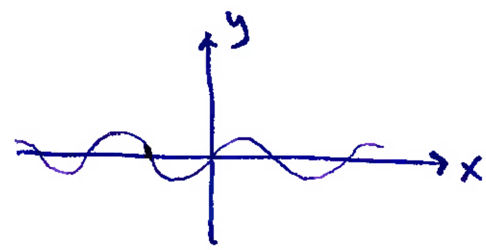
then we set $\boxed{y(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds}$

<u>example</u>

$f(x) = \sin(x)$

$g(x) = 0$

$a = 1$

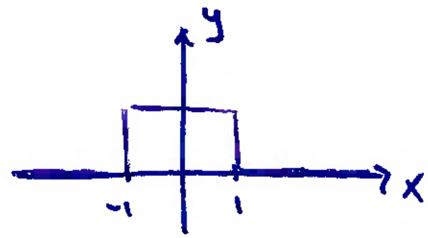


$$y(x,t) = \frac{1}{2}\left[\sin(x-t) + \sin(x+t)\right]$$

half of initial to each direction at speed of a

example

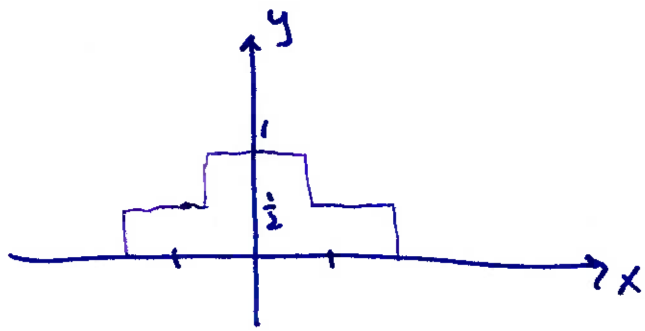
$$f(x) = \begin{cases} 1 & -1 < x < 1 \\ 0 & \text{else} \end{cases}$$

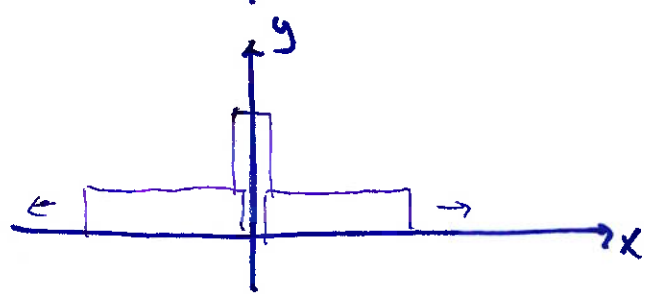


$g(x) = 0$

$a = 1$

$$y(x,t) = \frac{1}{2}\left[f(x-t) + f(x+t)\right]$$



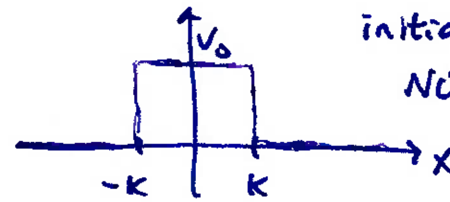


width is 2 (same as f(x))

example

$f(x) = 0$

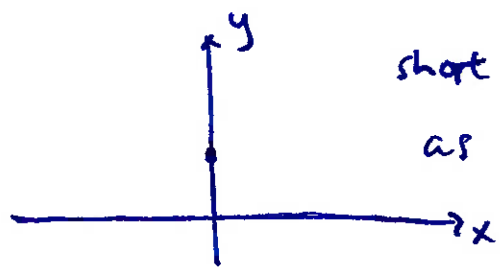
$$g(x) = \begin{cases} V_0 & \text{if} \quad |x| < K \\ 0 & \text{else} \end{cases}$$



initial velocity
NOT displacement

$$y(x,t) = \frac{1}{2a} \int_{x-at}^{x+at} g(s)\, ds$$

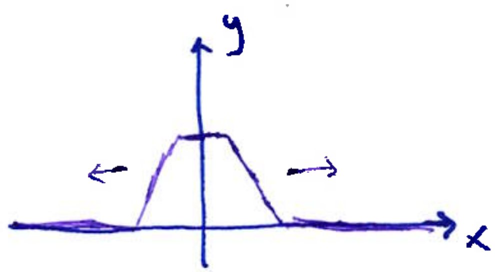
displacement is the accumulation under $g(x)$

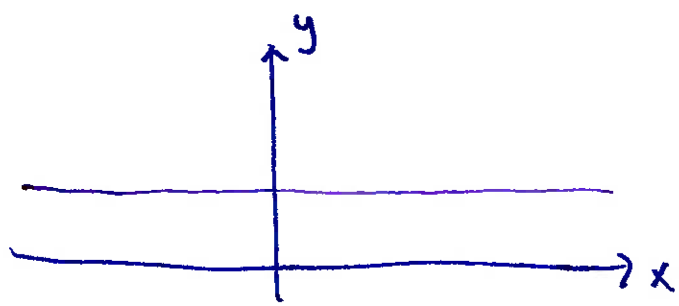


short time later

as t increases, center goes up because more area under $g(x)$ is accumulated

and the initial then propagates away





string does NOT come back to $y = 0$